1. Consider the matrix
$$A = \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix}$$
   (a) Calculate the eigenvalues and eigenvectors of A.
   (b) Calculate the change of basis matrix from the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$ to the basis of eigenvectors and verify that A is diagonal in this basis.
   (c) Verify that $\mathrm{Tr}(A) = \lambda_1 + \lambda_2$ and $\det A = \lambda_1\lambda_2$ where λ_1, λ_2 are the eigenvalues of A.

2. Consider the linear map,
$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad (U_1, U_2) \mapsto (U_1 + U_2, 3U_2, 4U_1).$$
   (a) Write down the matrix representation of T with repsect to the standard bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ for $\mathbb{R}^2$ and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ for $\mathbb{R}^3$.
   (b) Calculate the kernel and range of the linear map and verify the rank-nullity theorem in this case.

3. Let $g(u, v) = (uv, u - v, v^2u)$ and let $f(x, y, z) = (x + y, e^{x-y})$. Calculate the differentianls of g, f and of $f \circ g$ to confirm the chain rule for the differential of $f \circ g$.

4. Let
$$\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}, \quad C = \{x^2 + y^2 = 1, -1 \leq z \leq 1\}$$
be the unit 2-sphere, and the circular, unit cylinder with axis along the z direction respectively. Let
$$\varphi(x, y, z) = (\sqrt{1 - z^2}x, \sqrt{1 - z^2}y, z)$$
   (a) Show that φ maps C onto $\mathbb{S}^2$ and bijectively between $\{x^2 + y^2 = 1, -1 < z < 1\}$ and $\mathbb{S}^2 \backslash \{(0, 0, \pm 1)\}$.
   (b) Show that φ is an area-preserving map from C to $\mathbb{S}^2$.
   (c) Show that φ is not distance preserving. That is, in general, for γ a curve on C, the length $L(\gamma) \neq L(\varphi(\gamma))$.

   Hint: Use cylindrical polar coordinates.

5. Let $F = (-y, x, 0)$ and let $S = \{(x, y, 0) : x^2 + y^2 \leq 1\}$ be the unit disc in the $z = 0$ plane with boundary $C = \partial S$ parametrised by
$$C(t) = (\cos t, \sin t, 0), \quad 0 \leq t \leq 2\pi.$$
   (a) Directly calculate $\oint_C F \cdot d\mathbf{s}$ without using Green's theorem, Stokes' theorem or the Divergence Theorem.
   (b) Show that the unit normal to S is $N = (0, 0, 1)$.
   (c) Show that $\mathrm{curl}\, F = (0, 0, 2)$.
   (d) Directly calculate $\iint_S \mathrm{curl}\, F \cdot d\mathbf{A}$ without using Green's theorem, Stokes' theorem or the Divergence Theorem.
   (e) Let $S' = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ be the northern hemisphere. Using any method, calculate
$$\iint_{S'} \mathrm{curl}\, F \cdot d\mathbf{A}.$$

Solutions

1. Consider the matrix

$$A = \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix}$$

(a) Calculate the eigenvalues and eigenvectors of A.

The characteristic polynomial is

$$\det(A - \lambda\,\mathrm{Id}) = (-3 - \lambda)(-\lambda) - 4 = (\lambda - 1)(\lambda + 4)$$

hence

$$\lambda_1 = -4, \quad \lambda_2 = 1.$$

For V_1 the eigenvector associated to $\lambda_1 = -4$, $V_1 \in \ker(A + 4\,\mathrm{Id})$ and

$$A + 4\,\mathrm{Id} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

which has kernel spanned by $V_1 = (-2, 1)$ which can be seen for example since row-2 is twice row-1 and so $\ker(A+4\,\mathrm{Id})$ is the kernel of $(x, y) \mapsto x+2y$.
For V_2 the eigenvector associated to $\lambda_2 = 1$, $V_2 \in \ker(A - \mathrm{Id})$ and

$$A - \mathrm{Id} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

which has kernel spanned by $V_2 = (1, 2)$ which can be seen for example since row-1 is the negative of twice row-2 so $\ker(A - \mathrm{Id})$ is the kernel of $(x, y) \mapsto 2x - y$.
Thus

$$V_1 = a \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad V_2 = b \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

for any $a, b, \neq 0$.

(b) Calculate the change of basis matrix from the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$ to the basis of eigenvectors and verify that A is diagonal in this basis.

The change of basis is the inverse of P, the matrix of eigenvectors. So for example, taking $a = b = 1$,

$$P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}, \quad P^{-1} = -\frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}.$$

In the basis of eigenvectors, A becomes

$$D = P^{-1}AP = -\frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$$

(c) Verify that $\mathrm{Tr}(A) = \lambda_1 + \lambda_2$ and $\det A = \lambda_1\lambda_2$ where λ_1, λ_2 are the eigenvalues of A.

$$\mathrm{Tr}\, A = -3 + 0 = -4 + 1 = \lambda_1 + \lambda_2$$
$$\det A = (-3) \cdot 0 - 2 \cdot 2 = (-4) \cdot 1 = \lambda_1\lambda_2.$$

2. Consider the linear map,

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad (U_1, U_2) \mapsto (U_1 + U_2, 3U_2, 4U_1).$$

(a) Write down the matrix representation of T with repsect to the standard bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ for $\mathbb{R}^2$ and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ for $\mathbb{R}^3$.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 4 & 0 \end{pmatrix}$$

(b) Calculate the kernel and range of the linear map and verify the rank-nullity theorem in this case.

For the kernel, $U = (U_1, U_2) \in \ker A$ if and only if

$$U_1 + U_2 = 0, \quad 3U_2 = 0, \quad 4U_1 = 0$$

if and only if $U = (0, 0)$. Thus $\ker A = \{(0, 0)\}$.
The range is the span of

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

which we could write explicitly as

$$\{a_1V_1 + a_2V_2 : (a, b) \in \mathbb{R}^2\}.$$

Note that V_1, V_2 are linearly independent. Otherwise $V_1 = aV_2$ and looking at the second row, this implies that $0 = 3a$ hence $a = 0$, hence $V_1 = 0$ which is false. Thus $\mathrm{rnk}\, A = \dim \mathrm{rng}\, A = 2$.
Thus we have

$$\dim \ker A + \mathrm{rnk}\, A = 2 = \dim \mathrm{dom}\, A$$

verifying the rank-nullity theorem.

3. Let $g(u, v) = (uv, u - v, v^2 u)$ and let $f(x, y, z) = (x + y, e^{x-y})$. Calculate the differentianls of g, f and of $f \circ g$ to confirm the chain rule for the differential of $f \circ g$.

Directly calculating,

$$dg = \begin{pmatrix} \partial_u(uv) & \partial_v(uv) \\ \partial_u(u - v) & \partial_v(u - v) \\ \partial_u(v^2 u) & \partial_v(v^2 u) \end{pmatrix} = \begin{pmatrix} v & u \\ 1 & -1 \\ v^2 & 2uv \end{pmatrix}$$

and

$$df = \begin{pmatrix} \partial_x(x + y) & \partial_y(x + y) & \partial_z(x + y)) \\ \partial_x(e^{x-y}) & \partial_y(e^{x-y}) & \partial_z(e^{x-y}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ e^{x-y} & -e^{x-y} & 0 \end{pmatrix}$$

For the composition,

$$f \circ g(u, v) = f(uv, u - v, v^2 u) = (uv + u - v, e^{uv-u+v})$$

hence

$$\begin{aligned} d(f \circ g) &= \begin{pmatrix} \partial_u(uv + u - v) & \partial_v(uv + u - v) \\ \partial_u(e^{uv-u+v}) & \partial_v(e^{uv-u+v}) \end{pmatrix} \\ &= \begin{pmatrix} v + 1 & u - 1 \\ (v - 1)e^{uv-u+v} & (u + 1)e^{uv-u+v} \end{pmatrix} \end{aligned}$$

The chain rule gives $d(f \circ g)|_{(u,v)} = df|_{g(u,v)} \circ dg|_{(u,v)}$. Substituting $g(u, v)$ into df gives

$$df|_{(g(u,v))} = \begin{pmatrix} 1 & 1 & 0 \\ e^{uv-u+v} & -e^{uv-u+v} & 0 \end{pmatrix}$$

Then

$$\begin{aligned} df|_{g(u,v)} \circ dg|_{(u,v)} &= \begin{pmatrix} 1 & 1 & 0 \\ e^{uv-u+v} & -e^{uv-u+v} & 0 \end{pmatrix} \begin{pmatrix} v & u \\ 1 & -1 \\ v^2 & 2uv \end{pmatrix} \\ &= \begin{pmatrix} v + 1 & u - 1 \\ (v - 1)e^{uv-u+v} & (u + 1)e^{uv-u+v} \end{pmatrix} \\ &= d(f \circ g)|_{(u,v)} \end{aligned}$$

4. Let

$$\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}, \quad C = \{x^2 + y^2 = 1, -1 \leq z \leq 1\}$$

be the unit 2-sphere, and the circular, unit cylinder with axis along the z direction respectively. Let

$$\varphi(x, y, z) = (\sqrt{1 - z^2}x, \sqrt{1 - z^2}y, z)$$

(a) Show that φ maps C onto $\mathbb{S}^2$ and bijectively between $\{x^2 + y^2 = 1, -1 < z < 1\}$ and $\mathbb{S}^2 \backslash \{(0, 0, \pm 1)\}$.

For $(0, 0, \pm 1) \in \mathbb{S}^2$, given any (u, v) with $u^2 + v^2 = 1$ we have

$$\varphi(u, v, \pm 1) = (0, 0, \pm 1)$$

hence $(0, 0, \pm 1)$ is in the range of φ.
It then suffices to prove the second part. For convenience, let $\overset{\circ}{C} = \{x^2 + y^2 = 1, -1 < z < 1\}$. We need to show φ is a bijection between $\overset{\circ}{C}$ and $\mathbb{S}^2 \backslash \{(0, 0, \pm 1)\}$.
Let $(x, y, z) \in \mathbb{S}^2 \backslash \{(0, 0, \pm 1)\}$ and let $(u, v, w) = \left(\frac{x}{\sqrt{1-z^2}}, \frac{y}{\sqrt{1-z^2}}, z\right)$. Then

$$u^2 + v^2 = \frac{x^2 + y^2}{1 - z^2} = \frac{1 - z^2}{1 - z^2} = 1$$

since $x^2 + y^2 + z^2 = 1$. Also $w = z \in (-1, 1)$. Thus $(u, v, w) \in \overset{\circ}{C}$. Now,

$$\begin{aligned}
\varphi(u, v) &= \left(\sqrt{1 - w^2}u, \sqrt{1 - w^2}v, w\right) \\
&= \left(\sqrt{1 - z^2}\frac{x}{\sqrt{1 - z^2}}, \sqrt{1 - z^2}\frac{y}{\sqrt{1 - z^2}}, z\right) \\
&= (x, y, z)
\end{aligned}$$

hence φ restricted to $\overset{\circ}{C}$ is onto $\mathbb{S}^2 \backslash \{(0, 0, \pm 1)\}$.
Now suppose $(x, y, z), (u, v, w) \in \overset{\circ}{C}$ are such that $\varphi(x, y, z) = \varphi(u, v, w)$. Then $w = z$ and $\frac{u}{\sqrt{1-w^2}} = \frac{x}{\sqrt{1-z^2}}$ hence

$$u = \sqrt{1 - w^2}\frac{u}{\sqrt{1 - w^2}} = \sqrt{1 - z^2}\frac{x}{\sqrt{1 - z^2}} = x.$$

Likewise $v = y$. Thus φ is injective on $\overset{\circ}{C}$.

(b) Show that φ is an area-preserving map from C to $\mathbb{S}^2$.

Let $R \subseteq C$. We need to show that $\text{Area}(\varphi(R)) = \text{Area}(R)$. First notice that $\text{Area}(R) = \text{Area}(R \cap \mathring{C})$ since $\mathring{C}$ is only missing the boundary $\{x^2 + y^2 = 1, z = \pm 1\}$ which is the disjoint union of two curves, both of which have zero two-dimensional area. Likewise, $\text{Area}(\varphi(R)) = \text{Area}(\varphi(R \cap \mathring{C})$ since $\varphi(\mathring{C})$ only omits the two points $(0, 0, \pm 1)$ which also have zero two-dimensional area.

Thus we may restrict to $R \subseteq \mathring{C}$ on which φ is a bijection.
We use cylindrical coordinates for C:

$$F(r, \theta) = (\cos\theta, \sin\theta, r) : \quad -1 < r < 1, \quad 0 < \theta < 2\pi.$$

We have that F maps bijectively onto $\mathring{C} \backslash \{(1, 0, z) : -1 < z < 1\}$ and so only omits a line which has zero two-dimensional area. Letting $S = F^{-1}(R)$ we have

$$\begin{aligned}
\text{Area}(R) &= \iint_S |\partial_r F \times \partial_\theta F| \, dr d\theta \\
&= \iint_S |(0, 0, 1) \times (-\sin\theta, \cos\theta, 0)| \, dr d\theta \\
&= \iint_S dr d\theta.
\end{aligned}$$

On the other hand, $\varphi \circ F$ only omits the curve $\{(1, 0, \sqrt{1 - r^2}) : -1 < r < 1\}$ from $\mathbb{S}^2 \backslash \{(0, 0, \pm 1)\}$ which again has zero two-dimensional area. Computing as with C, but this time using

$$\varphi \circ F(r, \theta) = \left(\sqrt{1 - r^2} \cos\theta, \sqrt{1 - r^2} \sin\theta, r\right)$$

we get

$$\begin{aligned}
\partial_r(\varphi \circ F) &= \left(\frac{-r\cos\theta}{\sqrt{1 - r^2}}, \frac{-r\sin\theta}{\sqrt{1 - r^2}}, 1\right) \\
\partial_\theta(\varphi \circ F) &= \left(-\sqrt{1 - r^2} \sin\theta, \sqrt{1 - r^2} \cos\theta, 0\right)
\end{aligned}$$

hence

$$\begin{aligned}
\text{Area}(\varphi(R)) &= \iint_S |\partial_r(\varphi \circ F) \times \partial_\theta(\varphi \circ F)| \, dr d\theta \\
&= \iint_S \left| \left(-\sqrt{1 - r^2} \cos\theta, -\sqrt{1 - r^2} \sin\theta, -r \right) \right| dr d\theta \\
&= \iint_S dr d\theta.
\end{aligned}$$

Thus $\text{Area}(R) = \text{Area}(\varphi(R))$ and hence φ is area preserving.

(c) Show that φ is not distance preserving. That is, in general, for γ a curve on C, the length $L(\gamma) \neq L(\varphi(\gamma))$.

> Let $\gamma(t) = (1, 0, t) \in C$ for $t \in [-1, 1]$. Then
>
> $$L(\gamma) = \int_{-1}^{1} |\gamma'(t)|\, dt = \int_{-1}^{1} dt = 2.$$
>
> One the other hand, $\varphi \circ \gamma(t) = (\sqrt{1-t^2}, 0, t)$ hence
>
> $$L(\varphi(\gamma)) = \int_{-1}^{1} \left| \left(\frac{-t}{\sqrt{1-t^2}}, 0, 1 \right) \right| dt = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} dt = \pi.$$
>
> Thus for this particular γ, $L(\gamma) \neq L(\varphi(\gamma))$ hence φ is not distance preserving.

5. Let $F = (-y, x, 0)$ and let $S = \{(x, y, 0) : x^2 + y^2 \leq 1\}$ be the unit disc in the $z = 0$ plane with boundary $C = \partial S$ parametrised by

$$C(t) = (\cos t, \sin t, 0), \quad 0 \leq t \leq 2\pi.$$

(a) Directly calculate $\oint_C F \cdot d\mathbf{s}$ without using Green's theorem, Stokes' theorem or the Divergence Theorem.

> We have
>
> $$\begin{aligned} \int_c F \cdot d\mathbf{s} &= \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_0^{2\pi} dt = 2\pi \end{aligned}$$

(b) Show that the unit normal to S is $N = (0, 0, 1)$.

> Parametrising S by $\Phi(u, v) = (u, v, 0)$ we have $\mathbf{e}_u = (1, 0, 0)$ and $\mathbf{e}_v = (0, 1, 0)$ are a basis for the tangent space, hence $N = (0, 0, 1)$.

(c) Show that $\operatorname{curl} F = (0, 0, 2)$.

> $$\operatorname{curl} F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = (0, 0, 2)$$

(d) Directly calculate $\iint_S \operatorname{curl} F \cdot d\mathbf{A}$ without using Green's theorem, Stokes' theorem or the Divergence Theorem.

> $$\begin{aligned} \iint_S \operatorname{curl} F \cdot d\mathbf{A} &= \iint_S (0, 0, 1) \cdot (0, 0, 2) dA \\ &= \iint_S 2 dA = 2\pi. \end{aligned}$$

(e) Let $S' = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ be the northern hemisphere. Using any method, calculate

$$\iint_{S'} \operatorname{curl} F \cdot d\mathbf{A}.$$

By Stokes' theorem,

$$\iint_{S'} \operatorname{curl} F \cdot d\mathbf{A} = \int_C F \cdot d\mathbf{s} = \iint_S \operatorname{curl} F \cdot d\mathbf{A} = 2\pi$$