



Student number

Semester 1 Diagnostic Quiz, 2024

School of Mathematics and Statistics

MAST90103 Random Matrix Theory

Submission deadline:

This assignment consists of 9 pages (including this page)

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Question 1

Consider five matrices $A, B \in \mathbb{C}^{N \times N}$, $C \in \mathbb{C}^{N \times M}$, $D \in \mathbb{C}^{M \times N}$, and $E \in \mathbb{C}^{M \times M}$ of dimensions $N \times N$, $N \times M$, $M \times N$ and $M \times M$, respectively. Additionally, the matrix E should be invertible. Show the following three identities for the determinant.

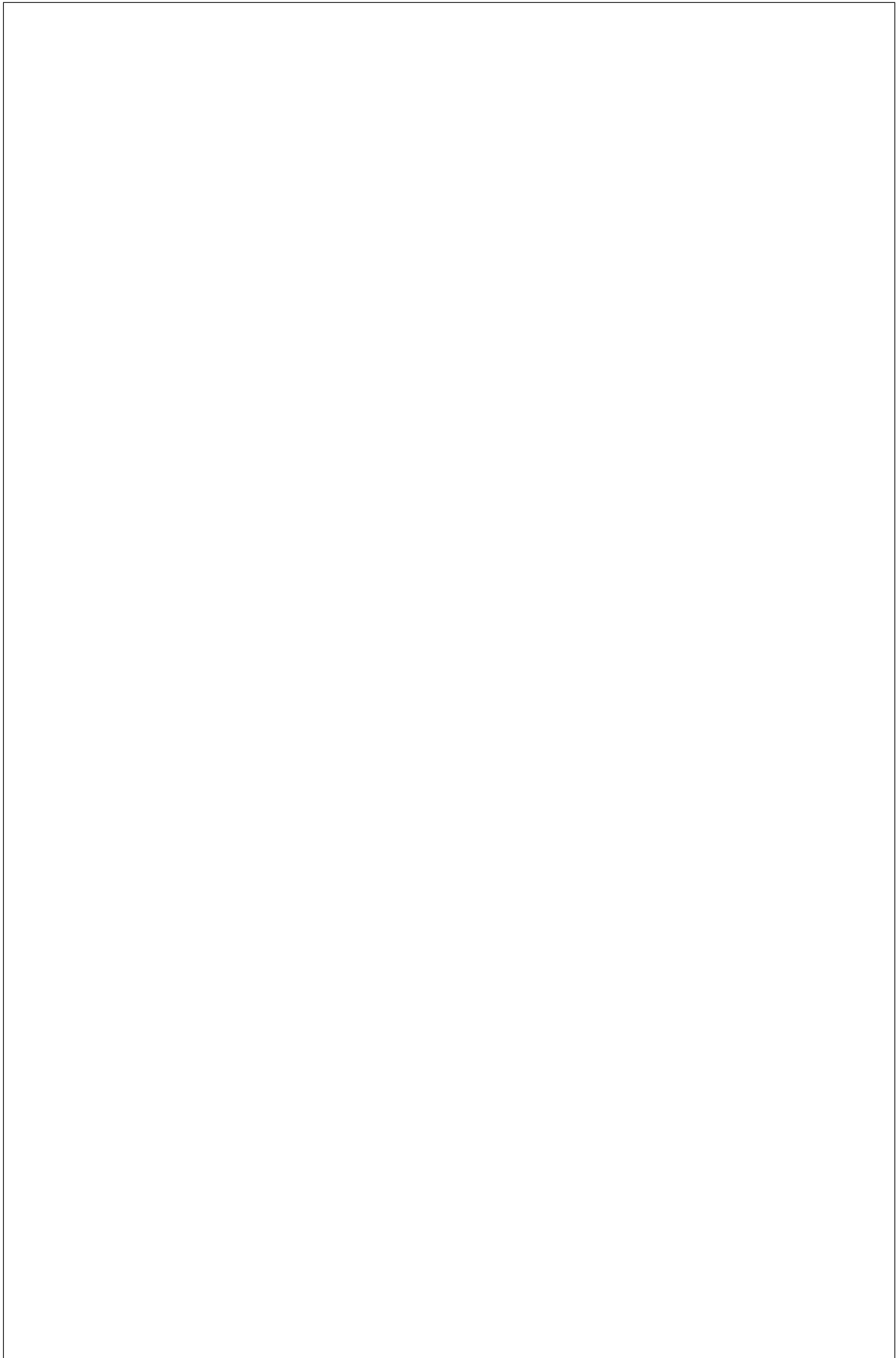
- (a) $\det[AB] = \det[A] \det[B]$, prove this by the Leibniz formula of the determinant

$$\det[A] = \sum_{\omega \in \mathbb{S}_N} \text{sign}(\omega) \prod_{j=1}^N A_{j\sigma(j)}$$

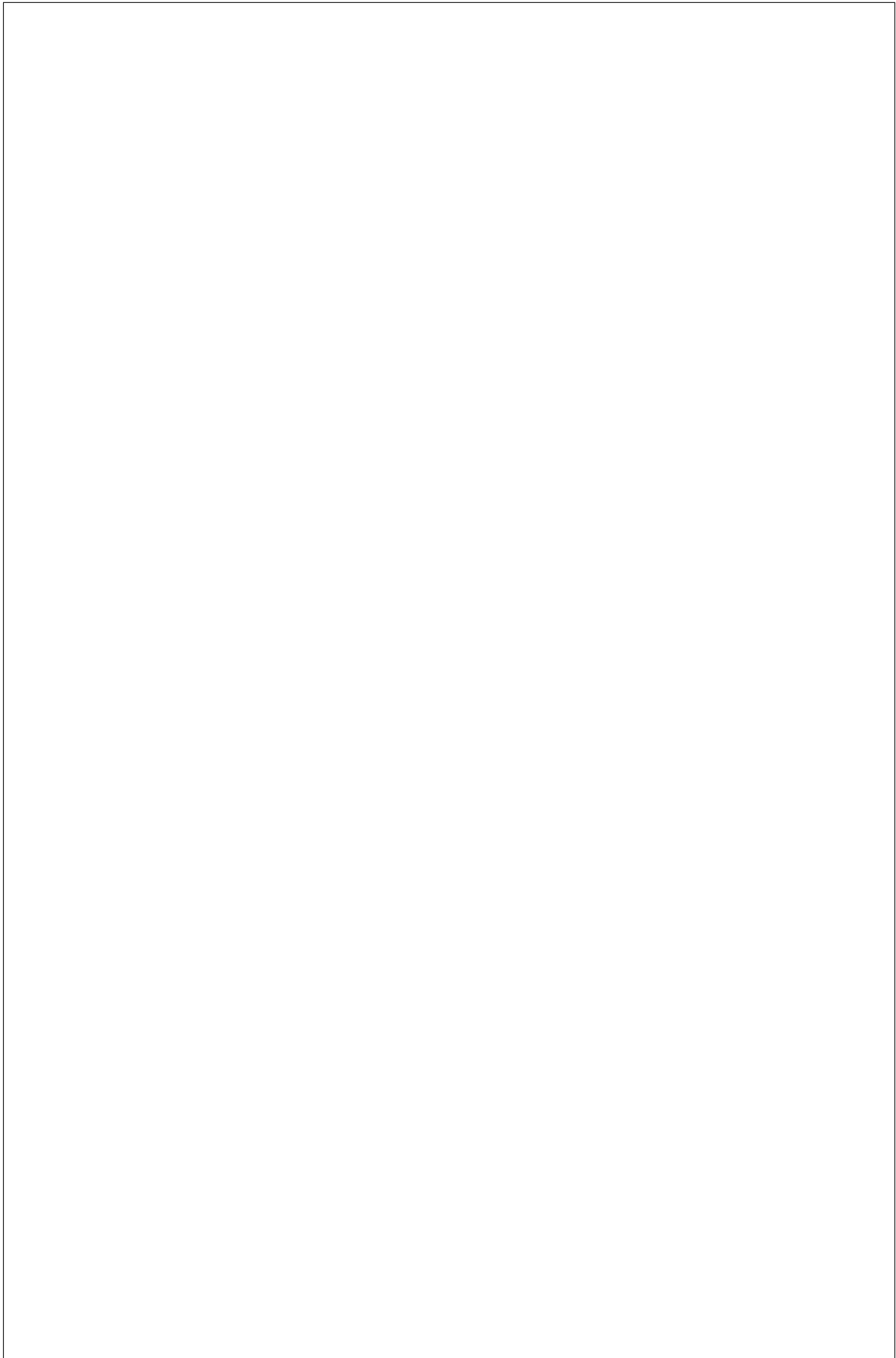
with \mathbb{S}_N the symmetric group comprising all permutations of N elements and $\text{sign}(\omega)$ is $+1$ when ω is an even permutation and -1 when it is an odd one;

- (b) $\det \begin{bmatrix} A & C \\ D & E \end{bmatrix} = \det[A - CE^{-1}D] \det[E]$, prove this by using identity (a);
- (c) $\det[\mathbf{1}_N - CD] = \det[\mathbf{1}_M - DC]$, prove this by using identity (b).

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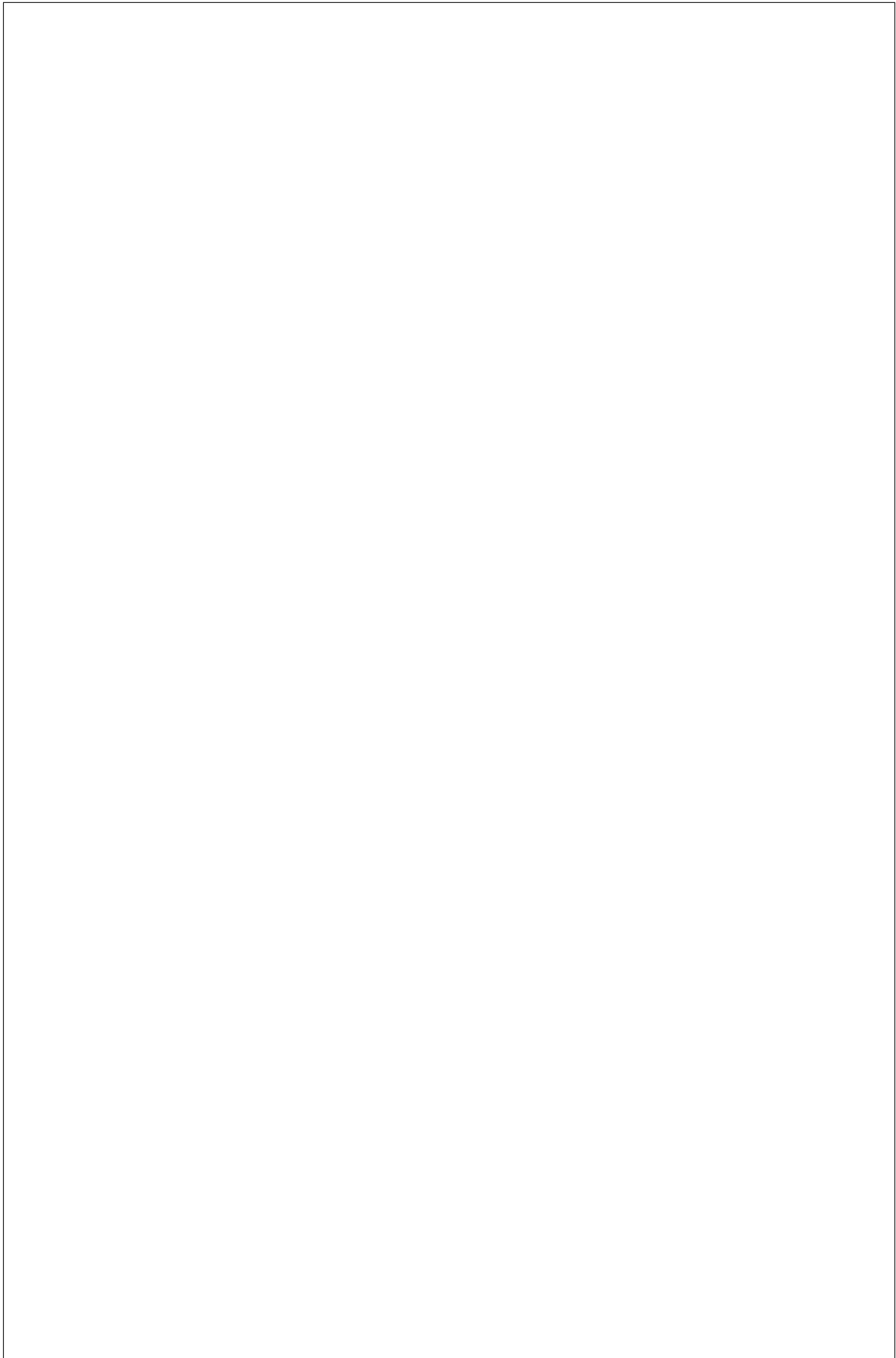


Question 2

Let $X = -X^T \in \mathbb{R}^{N \times N}$ be a real $N \times N$ antisymmetric matrix. Prove that

- (a) $\det[X] = 0$ whenever N is odd and show in this case that 0 is an eigenvalue of X ;
- (b) all eigenvalues are imaginary and come in complex conjugate pairs, meaning when λ is an eigenvalue then the complex conjugate $\lambda^* = -\lambda$ is also an eigenvalue.
- (c) if $v \in \mathbb{C}^N$ is an eigenvector of X to the eigenvalue λ , then v^* is an eigenvector of X to the eigenvalue $\lambda^* = -\lambda$.

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Question 3

- (a) Let $a, b \in \mathbb{C}$ be two fixed complex numbers and a has a positive real part $\operatorname{Re}(a) > 0$. Prove the following integral:

$$\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx] dx = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right],$$

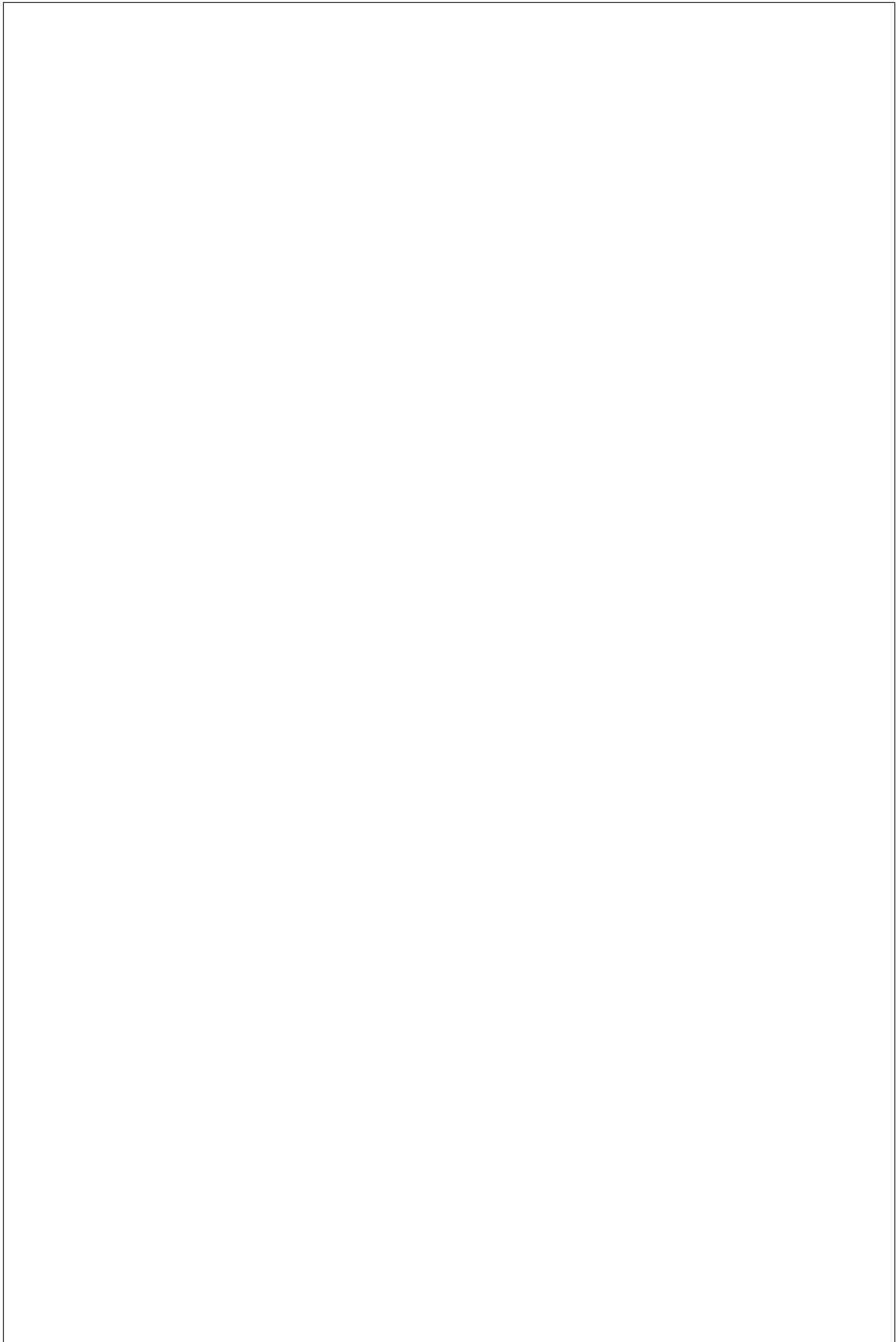
where \sqrt{a} is the principal value of the square root of a meaning it has a branch cut along the negative real line with the square root of a positive number being positive.

- (b) Let $A \in \mathbb{R}^{3 \times 3}$ be an invertible 3×3 real matrix. Compute the Gaussian integral

$$I(A) = \int_{\mathbb{R}^3} \exp[-x^T A x] d^3 x$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is a three dimensional column vector and the volume element is $d^3 x = dx_1 dx_2 dx_3$.

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End of Assignment

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