Student number

Semester 1 Diagnostic Quiz, 2024

School of Mathematics and Statistics

MAST90103 Random Matrix Theory

Submission deadline:

This assignment consists of 9 pages (including this page)

Question 1

Consider five matrices $A, B \in \mathbb{C}^{N\times N}$, $C \in \mathbb{C}^{N\times M}$, $D \in \mathbb{C}^{m\times N}$, and $E \in \mathbb{C}^{M\times M}$ of dimensions $N \times N$, $N \times M$, $M \times N$ and $M \times M$, respectively. Additionally, the matrix E should be invertible. Show the following three identities for the determinant.

(a) $\det[AB] = \det[A]\det[B]$, prove this by the Leibniz formula of the determinant

$$\det[A] = \sum\nolimits_{\omega\in\mathbb{S}_N} \mathrm{sign}(\omega) \prod\nolimits_{j=1}^{N} A_{j\sigma(j)}$$

with $\mathbb{S}_N$ the symmetric group comprising all permutations of N elements and $\mathrm{sign}(\omega)$ is $+1$ when ω is an even permutation and -1 when it is an odd one;

(b) $\det\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \det[A - CE^{-1}D]\det[E]$, prove this by using identity (a);

(c) $\det[\mathbf{1}_N - CD] = \det[\mathbf{1}_M - DC]$, prove this by using identity (b).

Question 2

Let $X = -X^T \in \mathbb{R}^{N \times N}$ be a real $N \times N$ antisymmetric matrix. Prove that

(a) $\det[X] = 0$ whenever N is odd and show in this case that 0 is an eigenvalue of X;

(b) all eigenvalues are imaginary and come in complex conjugate pairs, meaning when λ is an eigenvalue then the complex conjugate $\lambda^* = -\lambda$ is also an eigenvalue.

(c) if $v \in \mathbb{C}^N$ is an eigenvector of X to the eigenvalue λ, then v^* is an eigenvector of X to the eigenvalue $\lambda^* = -\lambda$.

Question 3

(a) Let $a, b \in \mathbb{C}$ be two fixed complex numbers and a has a positive real part $\mathrm{Re}(a) > 0$. Prove the following integral:

$$\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx]dx = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right],$$

where $\sqrt{a}$ is the principal value of the square root of a meaning it has a branch cut along the negative real line with the square root of a positive number being positive.

(b) Let $A \in \mathbb{R}^{3\times 3}$ be an invertible 3×3 real matrix. Compute the Gaussian integral

$$I(A) = \int_{\mathbb{R}^3} \exp[-x^T A^T A x] d^3x$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is a three dimensional column vector and the volume element is $d^3x = dx_1 dx_2 dx_3$.

End of Assignment