A Diffusive Epidemic Model with Free Boundaries

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1. Introduction

We determine the long-time dynamical behaviour of a reaction-diffusion system with free boundaries, which models the spreading of an epidemic whose moving front is represented by the free boundaries. We prove a spreading-vanishing dichotomy and determine exactly when each of the alternatives occurs. If the reproduction number $\mathbf{R_0}$ obtained from the corresponding ODE model is no larger than **1**, then the epidemic modelled here will vanish, while if $\mathbf{R_0 > 1}$, then the epidemic may vanish or spread depending on its initial size, determined by the dichotomy criteria. Moreover, when spreading happens, we show that the expanding front of the epidemic has an asymptotic spreading speed, which is determined by an associated semi-wave problem.

2. Research model

$$\begin{cases} u_t = d_1 u_{xx} - a_{11}u + a_{12}v, & (x,t)\in\Omega,\\ v_t = d_2 v_{xx} - a_{22}v + G(u), & (x,t)\in\Omega,\\ u(h(t),t) = v(h(t),t) = 0, & t>0,\\ u(g(t),t) = v(g(t),t) = 0, & t>0,\\ h'(t) = -\mu[u_x(h(t),t) + \rho v_x(h(t),t)], & t>0,\\ g'(t) = -\mu[u_x(g(t),t) + \rho v_x(g(t),t)], & t>0,\\ h(0)=h_0,\ g(0)=-h_0, & \\ u(x,0)=u_0(x), v(x,0)=v_0(x), & x\in\Omega_0, \end{cases}$$

where

$\Omega := \{x\in(g(t),h(t)), t\in(0,T]\}$,

$\Omega_0 := [-h_0,h_0]$.

3. Model explanation

- $\mathbf{u}$ the population density of bacteria,
- $\mathbf{v}$ infected human population,
- $\mathbf{a_{11}}$ natural death rate of bacteria,
- $\mathbf{a_{12}}$ growth rate of bacteria contributed by infective humans,
- $\mathbf{a_{22}}$ fatality rate of the infective human population,
- $\mathbf{G(u)}$ infective rate of humans.

$\mathbf{t}$ denotes time, $\mathbf{x=h(t)}$ and $\mathbf{g(t)}$ are the spreading fronts of this epidemic.

6. A semi-wave problem

For any parameter $\mathbf{c\geq 0}$, denoting $\mathbf{u(x,t)=W(x-ct)}$ and $\mathbf{v(x,t)=H(x-ct)}$, for $t>0$, then

$$\begin{cases} d_1W'' - cW' - a_{11}W + a_{12}H = 0,\\ d_2H'' - cH' - a_{22}H + G(W) = 0,\\ (W(0),H(0)) = (0,0),\\ (W(\infty),H(\infty)) = (u^*,v^*). \end{cases}$$

Theorem 4 (Monotone solution of semi-wave problem):

Assume that $R_0 > 1$. Then there exists a constant c^* such that for $c\geq c^*$, there is no monotone solution for the semi-wave problem, but for $c\in[0,c^*)$, this problem has a unique monotone solution $(W(s),H(s))$ in $C^2([0,\infty))\times C^2([0,\infty))$.

Lemma 5:

Let $(W_c(s),H_c(s))$ be the unique monotone solution of the semi-wave problem with $c\in[0,c^*)$, for any $\mu>0$ and $\rho>0$, there exists a unique $c_{\mu,\rho}\in[0,c^*)$ such that

$$\mu(\mathbf{W'_{c_{\mu,\rho}}(0)} + \rho\mathbf{H'_{c_{\mu,\rho}}(0)}) = \mathbf{c}_{\mu,\rho}.$$

4. Assumptions, Existence and uniqueness of the solution

Assumptions:

The infected rate $\mathbf{G(u)}$ satisfies

$(G1)$ $G\in C^1([0,\infty))$, $G(0)=0$, $G'(z)>0$ for $z\geq 0$;

$(G2)$ $\frac{G(z)}{z}$ is strictly decreasing and $\lim_{z\to+\infty}\frac{G(z)}{z} < \frac{a_{11}a_{22}}{a_{12}}$.

The initial value $(\mathbf{u_0},\mathbf{v_0})$ satisfies

$$\begin{cases} u_0\in C^2[-h_0,h_0], \quad u_0(\pm h_0)=0 \text{ and } u_0(x)>0 \text{ for } x\in(-h_0,h_0),\\ v_0\in C^2[-h_0,h_0], \quad v_0(\pm h_0)=0 \text{ and } v_0(x)>0 \text{ for } x\in(-h_0,h_0). \end{cases}$$

Theorem 1 (Existence and uniqueness):

The Problem of the research model admits a **unique positive** solution (u,v,h,g) defined for $t\in[0,\infty)$.

5. Spreading-Vanishing dichotomy and criteria

Theorem 2(Spreading-Vanishing dichotomy):

Assume that (u,v,h,g) is the solution of the research model, and denote

$$h_\infty := \lim_{t\to\infty} h(t) \text{ and } g_\infty := \lim_{t\to\infty} g(t).$$

Then one of the following cases must happen:

(i) **Vanishing**:

$$h_\infty - g_\infty < \infty \text{ and } \lim_{t\to\infty}(u(x,t),v(x,t)) = (0,0) \text{ uniformly for } x\in[g(t),h(t)];$$

(ii) **Spreading**:

$$h_\infty = -g_\infty = \infty \text{ and } \lim_{t\to\infty}(u(x,t),v(x,t)) = (u^*,v^*) \text{ locally uniformly in } \mathbb{R},$$

where $(\mathbf{u^*},\mathbf{v^*})$ is the unique positive stable-state of the corresponding ODE problem of the **Research model**.

The basic reproduction number:

$$\mathbf{R_0} := \frac{\mathbf{G'(0)a_{12}}}{\mathbf{a_{11}a_{22}}},$$

and criteria length:

$$\mathbf{L^*} := \frac{\pi}{\mathbf{2}}\sqrt{\frac{\mathbf{d_1a_{22}+d_2a_{11}+\sqrt{(d_1a_{22}+d_2a_{11})^2+4d_1d_2a_{11}a_{22}(R_0-1)}}}{\mathbf{2a_{11}a_{22}(R_0-1)}}}.$$

Theorem 3 (Spreading-Vanishing criteria):

Let (u,v,h,g) be the solution of the research model.

(i) If $R_0\leq 1$, then vanishing always occurs.

(ii) If $R_0>1$, then there exists a critical length $L^*>0$ independent of the initial data (u_0,v_0) such that if $h_0\geq L^*$, then spreading always happens, and if $h_0<L^*$, then there exists $\mu^*>0$ depending on (u_0,v_0) such that vanishing happens when $0<\mu\leq\mu^*$ and spreading happens when $\mu>\mu^*$.

7. The asymptotic spreading speed

Suppose $R_0>1$ and spreading happens to the research model, we have

$$\lim_{t\to\infty}\frac{\mathbf{h(t)}}{\mathbf{t}} = -\lim_{t\to\infty}\frac{\mathbf{g(t)}}{\mathbf{t}} = \mathbf{c}_{\mu,\rho}.$$